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ON THE NUMBER OF COPIES OF ONE HYPERGRAPH IN ANOTHER

BY

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ABSTRACT

Given two hypergraphs H and G, let $N(G,H)$ denote the number of subhypergraphs of G isomorphic to H. Let $N(l,H)$ denote the maximum of $N(G,H)$, taken over all G with exactly l edges. In [1] Noga Alon analyzes the asymptotic behaviour of $N(l,H)$ for H a graph. We generalize this to hypergraphs:

THEOREM: *For a hypergraph H with fractional cover number ρ^*, $N(G,H) = \theta(l^{\rho^*})$.*

The interesting part of this is the upper bound, which is shown to be a simple consequence of an entropy lemma of J. Shearer.

In a special case which includes graphs, we also provide a different proof using a hypercontractive estimate.

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1. The main theorem

For the purposes of this note, a **hypergraph** H is a collection of distinct subsets (**edges**) of some finite (**vertex**) set $V = V(H)$ (so we do not allow repeated edges). Throughout the paper G, H denote hypergraphs.

We write $N(G,H)$ for the number of (labelled) copies of H in G, and set

$$N(l,H) = \max\{N(G,H)\colon |G| = l\}.$$

We are interested in how this behaves for fixed H and large l. Before stating our result, we need to recall a little terminology.

A **stable set** of H is a subset of $V(H)$ meeting each edge of H at most once. A **fractional stable set** is a function $\phi\colon V \to [0,1]$ such that $\sum_{v\in e}\phi(v) \leq 1$ for each $e \in E$. The **fractional stable number** of H, denoted $\alpha^*(H)$, is the maximum over fractional stable sets ϕ of $\sum_{v\in V}\phi(v)$.

An (**edge**) **cover** of H is a set of edges whose union is $V(H)$. A **fractional cover** is $\psi\colon E \to [0,1]$ such that $\sum_{e\ni v}\psi(e) \geq 1$ for each $v \in V$. The **fractional cover number** of H, $\rho^*(H)$, is the minimum over fractional covers ψ of $\sum_{e\in E}\psi(e)$. By the duality theorem of linear programming one has $\rho^*(H) = \alpha^*(H)$ for every H.

We will show that for given H, the growth of $N(l,H)$ is basically a function of $\rho^*(H)$:

THEOREM 1.1: *For any hypergraph H, $N(l,H) = \theta(l^{\rho^*(H)})$.*

For graphs this was shown (using different language) in [1].

Proof: As mentioned above, the interesting part of the theorem is the upper bound. For the lower bound, let ϕ be an optimal fractional stable set of H (i.e. one maximizing $\sum\phi(v)$). Let $l_1 = l/|H|$, $V(H) = \{v_1 \dots v_k\}$, and for each $i \in [k]$, let X_i be a set of $l_1^{\phi(v_i)}$ vertices (with distinct X_i's disjoint and $\lfloor\ \rfloor$'s omitted). Set $V(G) = \bigcup X_i$.

Now let G consist of all edges $\{x_{i_1} \dots x_{i_s}\}$ with $x_{i_j} \in X_{i_j}$ and $\{v_{i_1} \dots v_{i_s}\} \in H$. Since ϕ is a fractional stable set, G contains at most l_1 copies of any edge of H, so at most l edges in all. On the other hand, G clearly contains at least $\prod |X_i| = l_1^{\rho^*(H)} = \Omega(l^{\rho^*(H)})$ copies of H.

We now turn to the upper bound. Here our main tool is the following lemma of J. Shearer.

For $\mathcal{W}$ a hypergraph on V and $F \subseteq V$, the **trace** of $\mathcal{W}$ on F is $\mathrm{Tr}(\mathcal{W},F) = \{W \cap F\colon W \in \mathcal{W}\}$.

LEMMA 1.2: *Suppose $\mathcal{W}$ is a hypergraph on a set V and $F_1 \ldots F_s$ are (not necessarily distinct) subsets of V such that each $x \in V$ belongs to at least t of the F_j's. Then*

$$|\mathcal{W}| \leq \prod_{j=1}^{s} |\mathrm{Tr}(\mathcal{W}, F_j)|^{1/t}.$$

(The lemma is stated in [4] with "exactly" in place of "at least," but this is easily seen to be equivalent.)

Let $N = N(G, H)$ and $V(H) = \{v_1 \ldots v_k\}$. We regard a copy of H in G as an injection $\sigma: V(H) \to V(G)$ such that for each $e \in H$, $\sigma(e)$ (defined in the obvious way) belongs to G, and write Σ for the set of such σ's.

PROPOSITION 1.3: *There is a partition $V(G) = V_1 \cup \cdots \cup V_k$ such that*

$$|\{\sigma \in \Sigma: \sigma(v_i) \in V_i \quad i = 1 \ldots k\}| \geq k^{-k} N. \tag{1}$$

Proof: For $\pi: V(G) \to [k]$ chosen uniformly at random, $V_i := \pi^{-1}(i)$ and $\sigma \in \Sigma$, we have $\Pr(\sigma(v_i) \in V_i \ \forall i) = k^{-k}$. Thus the right hand side of (1) is the expected value of the left hand side when the V_i's are chosen at random in this way. ∎

To prove the upper bound, it is thus enough to fix a partition $V(G) = V_1 \cup \cdots \cup V_k$ and bound the number, say M, of $\sigma \in \Sigma$ for which $\sigma(v_i) \in V_i$ for all i. Writing Σ' for the set of such σ, we may identify $\sigma \in \Sigma'$ with the subset $W_\sigma = \{\sigma(v_1) \ldots \sigma(v_k)\}$ of $V(G)$. We will apply Lemma 1.2 to the collection $\mathcal{W} = \{W_\sigma: \sigma \in \Sigma'\}$.

Clearing denominators in a rational-valued optimal fractional cover of H (note this always exists) gives integers s, t with $s/t = \rho^*$ and (not necessarily distinct) $A_1 \ldots A_s \in H$ such that each $v \in V(H)$ is in at least t of the A_j's. For $j = 1 \ldots s$, let $F_j = \bigcup_{v_i \in A_j} V_i \subseteq V(G)$, and set $\mathcal{F} = \{F_1 \ldots F_s\}$. Then each $x \in V(G)$ is in at least t members of $\mathcal{F}$.

Now (the point) for each $j \in [s]$ and $\sigma \in \Sigma'$, we have $W_\sigma \cap F_j \in G$, so that, crudely,

$$|\mathrm{Tr}(\mathcal{W}, F_j)| \leq l. \tag{2}$$

Lemma 1.2 now gives the desired bound:

$$M \leq \prod_{j=1}^{s} |\mathrm{Tr}(\mathcal{W}, F_j)|^{1/t} \leq l^{s/t} = l^{\rho^*}. \quad ∎$$

Remarks: One can improve this a bit by being more careful in (2). If we restrict to Σ', then each edge of G is a copy of just one edge of H. Then if ψ is an optimal fractional cover of H and $l(A)$ denotes the number of copies of A ($\in H$) in G (so $\sum \psi(A) = \rho^*$ and $l(A) = \gamma(A) \cdot l$ with $\sum \gamma(A) = 1$), then the above argument gives

$$M \leq \prod_{A \in H} l(A)^{\psi(A)} = \prod (\gamma(A) l)^{\psi(A)} \leq l^{\rho^*} \prod \left(\frac{\psi(A)}{\rho^*} \right)^{\psi(A)}.$$

In some cases the bound so obtained gives away practically nothing. For instance, if H is the complete graph K_k, then our bound is

$$k^k (\binom{k}{2}^{-1} l)^{k/2} \sim \sqrt{e}(2l)^{k/2}$$

whereas taking $G = K_q$ where $\binom{q}{2} = l$ (and ignoring questions of integrality), we have

$$N(G,H) = q(q-1)\cdots(q-k+1) \sim (2l)^{k/2}.$$

2. An upper bound via a hypercontractive estimate

In this section we present an upper bound on $N(l,H)$ using harmonic analysis. The bound is, unfortunately, generally weaker than that of Theorem 1.1 (though we do recover the latter in many cases, in particular in the original case of graphs). Still, the proof seems interesting enough (perhaps more interesting than the preceding one) to justify its inclusion here. It would be very nice if it could be extended to give the correct upper bound in general.

Given a hypergraph H an **even cover** of H is a list of (not necessarily distinct) edges of H such that each vertex of H belongs to a positive even number of edges in the list (counting multiplicities). For a hypergraph H, we write $\beta(H)$ for half the least cardinality of an even cover of H. Clearly $\beta(H) \geq \rho^*(H)$, and it is easy to see that the inequality is often strict.

That $\beta(H) = \rho^*(H)$ when H is a graph follows from a result of Lovász [6] stating that every graph has an optimal fractional cover taking values in $\{0, 1/2, 1\}$. It is easy to see that if ψ has minimum support among optimal fractional covers of this type, then $\sum_{e \ni x} \psi(e)$ is integral for each vertex x, so that a list of edges in which each e has multiplicity $2\psi(e)$ is an even cover of weight $2\rho^*(H)$.

Thus the next result agrees with Theorem 1.1 at least when H is a graph.

THEOREM 2.1: *For every fixed H, $N(l,H) = O(l^{\beta(H)})$.*

Proof: We write u_s for the character (Walsh function) corresponding to $s \in \mathbf{Z}_2^n$; that is, for $r \in \mathbf{Z}_2^n$, $u_s(r) = (-1)^{\langle s,r\rangle}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product. Given a hypergraph G with edges $E = \{e_1, \ldots, e_l\}$ and vertices $V = \{x_1, \ldots, x_n\}$, set $v_i = 1_{e_i}$ (i.e. $v_{ij} = 1_{\{x_j \in e_i\}}$) and $u = \sum_{i=1}^{l} u_{v_i}\colon \mathbf{Z}_2^n \to \mathbf{R}$.

Taking norms with respect to uniform measure on $\mathbf{Z}_2^n$, we will use the following generalization of Khinchine's inequality, whose proof, based on a hypercontractive estimate of Bonami and Beckner [2], [3], is given in [5]; see also [7]. (The **weight** of $s \in \{0,1\}^n$ is $\sum s_i$.)

LEMMA 2.2: *Let S be any set of vectors of weight at most k in $\{0,1\}^n$, and $u = \sum_{s \in S} u_s$. Then for any $q > 2$, $\|u\|_q \le (q-1)^{k/2}\|u\|_2$.*

For our particular u, notice that the orthonormality of the characters implies $\|u\|_2 = l^{1/2}$. Taking $\beta = \beta(H)$, $q = 2\beta$ and $c = (q-1)^{-\beta k}$, we thus have, from Lemma 2.2,

$$l^\beta = {\|u\|_2}^{2\beta} \ge c(\|u\|_{2\beta})^{2\beta} = c\int |u|^{2\beta}$$
$$\ge c\left|\int u^{2\beta}\right| = c\left|\int (\sum u_{e_i})^{2\beta}\right| = c\left|\sum_I \int \prod_{i \in I} u_{e_i}\right|,$$

where the last sum is over those I which are edge-lists of length 2β (so the products include multiplicities). But, again by orthonormality, $\int \prod_{i\in I} u_{e_i}$ is 1 if I is an even cover of $\bigcup_{i \in I} e_i$, and otherwise is 0. Thus $c^{-1} l^\beta$ is at least the number of even covers of cardinality 2β of subhypergraphs of G, which is clearly at least $N(G,H)$. ∎

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References

[1] N. Alon, *On the number of subgraphs of prescribed type of graphs with a given number of edges*, Israel Journal of Mathematics **38** (1981), 116–130.

[2] W. Beckner, *Inequalities in Fourier analysis*, Annals of Mathematics **102** (1975), 159–182.

[3] A. Bonami, *Etude des coefficients Fourier des fonctiones de $L^p(G)$*, Annales de l'Institut Fourier (Grenoble) **20** (1970), 335–402.

[4] F. R. K. Chung, P. Frankl, R. L. Graham and J. B. Shearer, *Some intersection theorems for ordered sets and graphs*, Journal of Combinatorial Theory, Series A **43** (1986), 23–37.

[5] R. F. Gundy, *Some Topics in Probability and Analysis*, AMS, Providence, 1989.

[6] L. Lovász, *2-matchings and 2-covers of hypergraphs*, Acta Mathematica Academiae Scientiarum Hungaricae **26** (1975), 433–444.

[7] M. Talagrand, *On Russo's approximate zero-one law*, The Annals of Probability **22** (1994), 1576–1587.